n-spin correlation functions for the two-dimensional Ising model*

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It is noted that expressions for the $T = T_c$ n-spin correlation functions of a two-dimensional Ising system obtained by continuum methods do agree with exact results for restricted geometries. This leads to the conjecture that the continuum result is true for all n and all geometries. Speculation regarding the behavior of these functions away from $T = T_c$ are made.

I. INTRODUCTION

Although the two-spin correlation function for the two-dimensional Ising model has been known for some time,[1,2] such a knowledge of n-spin correlation functions is not available. A formal expression for such functions away from the critical point has been obtained by McCoy, Tracy, and Wu.[3] It is not immediately obvious how to apply their results to $T = T_c$. At $T = T_c$ exact results are known for restrictive geometries. Kadanoff and Ceva[4] and Pink[4] found the n-point function for the collinear points along one of the lattice directions. This result has been extended by Au-Yang[5] to collinear points along a diagonal. For spin separations larger than the lattice constant the above results are

$$\langle \sigma(\vec{r}_1) \cdots \sigma(\vec{r}_n)\rangle_{T=T_c} = (\text{const}) \prod_{i<j}^{n} |\vec{r}_j - \vec{r}_i|^{(-1)^{i+j}/4} \quad . \tag{1}$$

Recently, Au-Yang[6] obtained an expression for the $T = T_c$ correlation function for two noncollinear but parallel pairs.

An alternate approach has been to work directly in the continuum limit using the analogy of the Ising problem to a two dimensional fermion field theory.[7-11] Luther and Peschel[8] gave the result for an arbitrary set of n spins located at $\vec{r}_i$:

$$\langle \sigma(\vec{r}_1) \cdots \sigma(\vec{r}_n)\rangle^2_{T=T_c} = (\text{const}) \sum_{\eta_i=\pm 1}{}' \prod_{i\neq j}^{n} |\vec{r}_j - \vec{r}_i|^{\eta_i\eta_j/2} \quad . \tag{2}$$

The prime in the summation indicates that the configurations of the η_i's are restricted to the ones satisfying $\sum \eta_i = 0$. Note that Eq. (2) is for the square of the correlation function. The same expression was obtained in Ref. 9 for collinear points and using similar techniques for noncollinear points.[11]

At first glance it appears that Eqs. (1) and (2) do not agree. Equation (2) looks much more symmetric. We wish to point out the embarrassingly simple fact that they *do agree.* For $n = 4$ it is a matter of trivial algebra. The proof for $n > 4$ will be given in Sec. II.

Finally, we have compared Eq. (2) for the case $n = 4$ to the results for a parallel set of points presented in Ref. 6. The two expressions agree to the order the results are stated in Ref. 6. These agreements lend strong support to the conjecture that Eq. (2) for the square of the correlation function is valid for all n and all geometric configurations.

The continuum techniques,[9,11] likewise, provide an expression for the n point function away from T_c. We conjecture that in the critical region

$$\langle \sigma(\vec{r}_1) \cdots \sigma(\vec{r}_n)\rangle^2_T = (\text{const}) \langle 0| \prod_{i=1}^{n} :\sin\sqrt{\pi}\varphi(\vec{r}_i): |0\rangle \quad , \tag{3}$$

where the expectation of the right-hand side is taken in the vacuum of a sine-Gordon field theory with a Lagrangian

$$L = \tfrac{1}{2}(\partial\varphi)^2 - m/\pi :\cos 2\sqrt{\pi}\varphi: \quad , \tag{4}$$

with $m = \frac{1}{4}(T - T_c)$. The above may serve as a basis for a mass perturbation away from $T = T_c$.

II. PROOF

We wish to show that Eq. (2) and the square of Eq. (1) are the same for the geometric situation where all the $\vec{r}_i$'s are collinear. As noted above, the observation for $n = 4$ requires the trivial algebra of bringing Eq. (2) to a common denominator. For $n > 4$,[12] the argument we found is somewhat baroque.

We study the *fourth* power of the correlation function. From Eq. (1) we find

$$\langle \sigma(x_1)\sigma(x_2) \cdots \sigma(x_n)\rangle^4 = (\text{const}) \prod_{i<j}^{n} (x_j - x_i)^{(-1)^{i+j}} \quad , \tag{5}$$

where the x_i's are placed in increasing order. This is, however, the expression for the vacuum expectation value of a product of free fermion fields[13] and thus can be expressed as a sum of products of two-body propagators:

$$\langle \sigma(x_1)\sigma(x_2) \cdots \sigma(x_n)\rangle^4 = |\langle 0|\psi(x_1)\psi^\dagger(x_2)\psi(x_3) \cdots \psi^\dagger(x_n)|0\rangle| = \sum \frac{1}{x_1-x_2}\frac{1}{x_3-x_4} \cdots \quad . \tag{6}$$

The square of Eq. (2) is likewise a sum of such terms. The proof of the equality of the two expressions proceeds by induction. We note that both Eqs. (1) and (2) are invariant under separate permutations of the x_i's with odd or even i's. At this stage we use factorization. Take any product of fermion propagators that appears in the decomposition of the square of Eq. (2). First, we would like to rule out terms with propagators $1/(x_i - x_j)$, with both i and j simultaneously even or odd, as such terms do not occur in Eq. (6). Should such a term occur we could permute it to involve x_i's for $i \le 4$. If we now separate the first four x's from the rest, factorization would imply that such a term would by present for correlation functions involving 4 or $n-4$ spins. Similarly the presence of the allowed terms can be checked using factorization and induction.

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